KAMPALA INTERNATIONAL UNIVERSITY

KIU
Journal of Science, Engineering and Technology

KJSET

Research Article

Direct Solution of Second Order Ordinary Differential Equations with a One-Step Hybrid Numerical Model

[1*]Ogunware, B. G., [2]Okafor, F. M., [3]Omole, E. O. and [4]Awoyemi, F. C.

[1, 3]Mathematics and Statistics Department, Joseph Ayo Babalola University, Ikeji - Arakeji, Osun State, Nigeria.
[2]National Mathematical Centre, Mathematics Programme, Abuja, Nigeria.
[4]Mathematics Department, Federal University of Technology, Akure, Ondo State, Nigeria.
Correspondence E-mail Address: ogunwaregbenga@gmail.com

Corresponding Author: Ogunware, B. G. (ogunwaregbenga@gmail.com)

Paper history:
Received 15 February 2023
Accepted in revised form 12 April 2023

Keywords
Second order ODE, Power series, Interpolation, Collocation, Hybrid block method, Linear multistep method.

Abstract

This study shows how to solve general second order ordinary differential equations directly using a one-step hybrid block technique. By means of collocation and interpolation of power series approximation, a continuous linear multistep method was developed. The developed method was evaluated at several grid and off-grid points to generate a discrete block approach. Order, error-constant, zero stability, consistency, and convergence were all investigated as basic features of the approach. The method was found to be more efficient and produce better approximations than previous approaches for solving some second order ordinary differential equations with initial value problems.

Nomenclature and units

τ	Collocation points
μ	Interpolation points

1.0 Introduction

This article presents the numerical solution of general second order initial value problems (IVP) of ordinary differential equations (ODEs) of the form:

$$y'' = f(x, y(x), y'(x)), y(x_0) = y_0, y'(x_0) = y'_0. \qquad (1)$$

Because a number of problems of the form of (1) are difficult to solve theoretically, approximate numerical integrations are routinely used to solve them (1). We often transform these to equivalent first-order ordinary differential equations and solve them using the appropriate method. (Lambert, 1973; Fatunla, 1988; Majid et al., 2006; Ogunware & Omole, 2020) are among scholars who have discussed the reduction technique. Some authors proposed a linear multistep technique for directly solving equation (1) to avoid the difficulty of transforming it to an equal system of first order ODEs. Such authors include (Awoyemi, 2001; Adesanya et al., 2008; Badmus and Yahaya, 2009; Olanegan et al., 2021). According to (Awoyemi, 2001), in terms of error estimation, the continuous linear multistep method outperforms the discrete method, providing a simplified coefficient for more analytical work at various points and ensuring easy appropriation of solution at all interior points of the integration interval. Continuous linear multistep approaches have been proposed by authors such as (Onumanyi et al., 1994; Okunnuga, 2008; Omar & Kuboye; 2015), to mention a few. To obtain beginning values for their approaches, these authors used the predictor-corrector, block methodology, and Taylor series expansion. According to (Adesanya, 2011), the predictor-corrector method is expensive because subroutines are difficult to build due to the unique procedures required to give starting values and change the step size, resulting in longer computer time and more human labor. The correctors are not in the same order as the predictors that were developed. As a result, the method's accuracy suffers. A number of authors, including (Olanegan et al., 2015; Kuboye et al., 2022; Ogunware et al., 2021; Adoghe et al., 2016), developed the hybrid technique. While this hybrid approach retains some of the properties of the continuous linear multistep method, it also shares the property of utilizing data from points other than the step point with Runge-Kutta methods (s). This strategy is beneficial for lowering the step number of a scheme while maintaining zero stability. Because the predictor-corrector approach fails to meet the above criteria, another method must be developed to compensate for the shortcomings. As a result, researchers devised the block technique to address the predictor-corrector method's drawbacks. (Adesanya, 2011), (Jator & Li, 2009) and (Omole & Ogunware, 2018) are some of such authors. (Abolarin, et al 2020a) recently considered developing and implementing a three-step hybrid block technique for directly solving linear and nonlinear second order ODEs.

The aim of this paper is the development and implementation of a new continuous single step hybrid block linear multistep method with six-off steps that is zero-stable, consistent, and convergent for direct and accurate solution of second order ODEs of initial value problems. The objectives of the research are to:

(a) adopt Power series as the basis function;
(b) develop a one-step hybrid linear multistep methods with continuous coefficients for second order initial value problems;
(c) get additional schemes from (b) that are needed to form the block.
(d) analyse the basic properties of the developed methods i.e order, error constant, consistency, zero stability, convergence and region of absolute stability;
(e) implement and adopt the developed scheme on sample second order initial value problems.

compare the numerical results with that of existing authors.

2.0 Materials and Methods

We consider power series as an approximate solution to the general second order ODEs initial value problems of the form (1) to be

$$y(x) = \sum_{j=0}^{\tau+\mu} a_j x^j. \qquad (2)$$

Where a_j's are the parameters to be determined, τ and μ are distinct number of collocation and interpolation points.

The second derivative of (2) is obtained as

$$y''(x) = \sum_{j=2}^{\tau+\mu} j(j-1)a_j x^{j-2}. \qquad (3)$$

The combination of equations (1) and (3) gives the differential system below:

$$y''(x) = \sum_{j=2}^{\tau+\mu} j(j-1)a_j x^{j-2} = f(x, y, y'). \qquad (4)$$

Collocating (4) at $x_{n+j}, j = 0\left(\frac{1}{7}\right)1$ and interpolating (2) at $x_{n+j}, j = \frac{3}{7}$ and $\frac{4}{7}$ gives a structure of non-linear equation of the form

$$\begin{bmatrix} 1 & x_{n+\frac{3}{7}} & x^2_{n+\frac{3}{7}} & x^3_{n+\frac{3}{7}} & x^4_{n+\frac{3}{7}} & x^5_{n+\frac{3}{7}} & x^6_{n+\frac{3}{7}} & x^7_{n+\frac{3}{7}} & x^8_{n+\frac{3}{7}} & x^9_{n+\frac{3}{7}} \\ 1 & x_{n+\frac{4}{7}} & x^2_{n+\frac{4}{7}} & x^3_{n+\frac{4}{7}} & x^4_{n+\frac{4}{7}} & x^5_{n+\frac{4}{7}} & x^6_{n+\frac{4}{7}} & x^7_{n+\frac{4}{7}} & x^8_{n+\frac{4}{7}} & x^9_{n+\frac{4}{7}} \\ 0 & 0 & 2 & 6x_n & 12x^2_n & 20x^3_n & 30x^4_n & 42x^5_n & 56x^6_n & 72x^7_n \\ 0 & 0 & 2 & 6x_{n+\frac{1}{7}} & 12x^2_{n+\frac{1}{7}} & 20x^3_{n+\frac{1}{7}} & 30x^4_{n+\frac{1}{7}} & 42x^5_{n+\frac{1}{7}} & 56x^6_{n+\frac{1}{7}} & 72x^7_{n+\frac{1}{7}} \\ 0 & 0 & 2 & 6x_{n+\frac{2}{7}} & 12x^2_{n+\frac{2}{7}} & 20x^3_{n+\frac{2}{7}} & 30x^4_{n+\frac{2}{7}} & 42x^5_{n+\frac{2}{7}} & 56x^6_{n+\frac{2}{7}} & 72x^7_{n+\frac{2}{7}} \\ 0 & 0 & 2 & 6x_{n+\frac{3}{7}} & 12x^2_{n+\frac{3}{7}} & 20x^3_{n+\frac{3}{7}} & 30x^4_{n+\frac{3}{7}} & 42x^5_{n+\frac{3}{7}} & 56x^6_{n+\frac{3}{7}} & 72x^7_{n+\frac{3}{7}} \\ 0 & 0 & 2 & 6x_{n+\frac{4}{7}} & 12x^2_{n+\frac{4}{7}} & 20x^3_{n+\frac{4}{7}} & 30x^4_{n+\frac{4}{7}} & 42x^5_{n+\frac{4}{7}} & 56x^6_{n+\frac{4}{7}} & 72x^7_{n+\frac{4}{7}} \\ 0 & 0 & 2 & 6x_{n+\frac{5}{7}} & 12x^2_{n+\frac{5}{7}} & 20x^3_{n+\frac{5}{7}} & 30x^4_{n+\frac{5}{7}} & 42x^5_{n+\frac{5}{7}} & 56x^6_{n+\frac{5}{7}} & 72x^8_{n+\frac{5}{7}} \\ 0 & 0 & 2 & 6x_{n+\frac{6}{7}} & 12x^2_{n+\frac{6}{7}} & 20x^3_{n+\frac{6}{7}} & 30x^4_{n+\frac{6}{7}} & 42x^5_{n+\frac{6}{7}} & 56x^6_{n+\frac{6}{7}} & 72x^7_{n+\frac{6}{7}} \\ 0 & 0 & 2 & 6x_{n+1} & 12x^2_{n+1} & 20x^3_{n+1} & 30x^4_{n+1} & 42x^5_{n+1} & 56x^6_{n+1} & 72x^7_{n+1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{bmatrix} = \begin{bmatrix} y_{n+3/7} \\ y_{n+4/7} \\ f_n \\ f_{n+1/7} \\ f_{n+2/7} \\ f_{n+3/7} \\ f_{n+4/7} \\ f_{n+5/7} \\ f_{n+6/7} \\ f_{n+1} \end{bmatrix}. \quad (5)$$

Solving for $a_j, j = 0(1)9$ in (5) using Gaussian elimination method and substituting into (2) gives a linear multistep method with continuous coefficients in the form:

$$y(t) = \alpha_{\frac{3}{7}}(t) y_{n+\frac{3}{7}} + \alpha_{\frac{4}{7}}(t) y_{n+\frac{4}{7}} + h^2 \begin{bmatrix} \beta_0(t) f_n + \beta_{\frac{1}{7}}(t) f_{n+\frac{1}{7}} + \beta_{\frac{2}{7}}(t) f_{n+\frac{2}{7}} + \beta_{\frac{3}{7}}(t) f_{n+\frac{3}{7}} \\ + \beta_{\frac{4}{7}}(t) f_{n+\frac{4}{7}} + \beta_{\frac{5}{7}}(t) f_{n+\frac{5}{7}} + \beta_{\frac{6}{7}}(t) f_{n+\frac{6}{7}} + \beta_1(t) f_{n+1} \end{bmatrix}. \quad (6)$$

Then, for easy simplification, using the transformation:

$$t = \frac{x - x_{n+k-1}}{h} \text{ and } \frac{dt}{dx} = \frac{1}{h}$$

The coefficients of y_{n+j} *and* f_{n+j} are obtained in terms of t as follows:

$$\alpha_{\frac{3}{7}}(t) = (-7t + 4)$$

$$\alpha_{\frac{4}{7}}(t) = (7t - 3)$$

$$\beta_0(t) = \left(\frac{22981}{2160}t^4 - \frac{121}{40}t^3 - \frac{535153}{12700800}t - \frac{331681}{14400}t^5 + \frac{1680}{540}t^6 - \frac{55223}{2160}t^7 + \frac{16807}{1440}t^8 - \frac{117649}{51840}t^9 + \frac{1}{2}t^2 + \frac{1919}{1481760} \right)$$

$$\beta_{\frac{1}{7}}(t) = \left(-\frac{1383307}{6350400}t - \frac{49}{6}t^3 + \frac{109417}{900}t^5 - \frac{88837}{480}t^6 + \frac{141659}{864}t^7 - \frac{50421}{640}t^8 + \frac{823543}{51840}t^9 - \frac{10927}{240}t^4 + \frac{3265}{148176} \right)$$

$$\beta_{\frac{2}{7}}(t) = \left(-\frac{160501}{4233600}t - \frac{49}{4}t^3 - \frac{1347647}{4800}t^5 + \frac{170471}{360}t^6 - \frac{7203}{16}t^7 + \frac{218491}{960}t^8 - \frac{823543}{17280}t^9 + \frac{14357}{160}t^4 - \frac{3775}{98784} \right)$$

$$\beta_{\frac{3}{7}}(t) = \left(-\frac{30757}{127008}t - \frac{46501}{432}t^4 + \frac{133427}{360}t^5 + \frac{2926819}{4320}t^6 + \frac{593047}{864}t^7 - \frac{420175}{1152}t^8 + \frac{823543}{10368}t^9 + \frac{245}{18}t^3 + \frac{21011}{370440} \right)$$

$$\beta_{\frac{4}{7}}(t) = \left(\frac{2009}{24}t^4 - \frac{174587}{576}t^5 + \frac{26411}{45}t^6 - \frac{271313}{432}t^7 + \frac{16807}{48}t^8 - \frac{823543}{10368}t^9 + \frac{175757}{2540160}t - \frac{245}{24}t^3 + \frac{5329}{1481760} \right)$$

$$\beta_{\frac{5}{7}}(t) = \left(-\frac{3283}{80}t^4 - \frac{49}{10}t^3 + \frac{22981}{150}t^5 - \frac{88837}{288}t^6 + \frac{55223}{160}t^7 - \frac{386561}{1920}t^8 + \frac{823543}{17280}t^9 - \frac{81817}{2116800}t + \frac{43}{49392} \right)$$

$$\beta_{\frac{6}{7}}(t) = \left(\frac{137519}{12700800}t - \frac{634207}{14400}t^5 + \frac{98441}{1080}t^6 - \frac{45619}{432}t^7 + \frac{184877}{2880}t^8 - \frac{823543}{51840}t^9 - \frac{49}{36}t^3 + \frac{49931}{4320}t^4 - \frac{481}{1481760} \right)$$

$$\beta_1(t) = \left(\frac{1}{6}t^3 + \frac{9947}{1800}t^5 - \frac{16807}{1440}t^6 + \frac{12005}{864}t^7 - \frac{16807}{1920}t^8 + \frac{117649}{51840}t^9 - \frac{343}{240}t^4 - \frac{263}{198450}t + \frac{2}{46305} \right)$$

Evaluating the continuous method at the non-interpolation points gives the following schemes

$$y_{n+1} = 4y_{n+\frac{4}{7}} - 3y_{n+\frac{3}{7}} + h^2 \left[\frac{1919}{1481760} f_{n+1} - \frac{481}{1481760} f_{n+\frac{1}{7}} + \frac{43}{49392} f_{n+\frac{2}{7}} + \frac{21011}{370440} f_{n+\frac{4}{7}} + \frac{3775}{98784} f_{n+\frac{5}{7}} + \frac{3265}{148176} f_{n+\frac{6}{7}} + \frac{2}{46305} f_n \right]. \quad (7)$$

$$y_{n+\frac{1}{7}} = 3y_{n+\frac{3}{7}} - 2y_{n+\frac{4}{7}} + h^2 \left[-\frac{31}{2963520} f_{n+1} + \frac{4769}{2963520} f_{n+\frac{1}{7}} + \frac{2101}{98784} f_{n+\frac{2}{7}} + \frac{100879}{2963520} f_{n+\frac{3}{7}} + \frac{3581}{740880} f_{n+\frac{4}{7}} - \frac{571}{987840} f_{n+\frac{5}{7}} + \frac{31}{296352} f_{n+\frac{6}{7}} - \frac{2}{46305} f_n \right]. \quad (8)$$

$$y_{n+\frac{2}{7}} = 2y_{n+\frac{3}{7}} - y_{n+\frac{4}{7}} + h^2 \left[-\frac{73}{493920} f_{n+\frac{1}{7}} + \frac{2171}{987840} f_{n+\frac{2}{7}} + \frac{12067}{740880} f_{n+\frac{3}{7}} + \frac{2171}{987840} f_{n+\frac{4}{7}} - \frac{73}{493920} f_{n+\frac{5}{7}} + \frac{31}{2963520} f_{n+\frac{6}{7}} + \frac{31}{2963520} f_n \right]. \quad (9)$$

$$y_{n+\frac{5}{7}} = 2y_{n+\frac{4}{7}} - y_{n+\frac{3}{7}} + h^2 \left[\frac{31}{2963520} f_{n+1} + \frac{31}{29633520} f_{n+\frac{1}{7}} - \frac{73}{493920} f_{n+\frac{2}{7}} + \frac{2171}{987840} f_{n+\frac{3}{7}} + \frac{12067}{740880} f_{n+\frac{4}{7}} + \frac{2171}{987840} f_{n+\frac{5}{7}} - \frac{73}{493920} f_{n+\frac{6}{7}} \right]. \quad (10)$$

$$y_{n+\frac{6}{7}} = 3y_{n+\frac{4}{7}} - 2y_{n+\frac{3}{7}} + h^2 \left[-\frac{2}{46305} f_{n+1} + \frac{31}{296352} f_{n+\frac{1}{7}} - \frac{571}{987840} f_{n+\frac{2}{7}} + \frac{3581}{740880} f_{n+\frac{3}{7}} + \frac{100879}{2963520} f_{n+\frac{4}{7}} + \frac{2101}{98784} f_{n+\frac{5}{7}} + \frac{4769}{2963520} f_{n+\frac{6}{7}} - \frac{31}{2963520} f_n \right]. \quad (11)$$

$$y_n = 4y_{n+\frac{3}{7}} - 3y_{n+\frac{4}{7}} + h^2 \left[\frac{1919}{1481760} f_n + \frac{2}{46305} f_{n+1} + \frac{3265}{148176} f_{n+\frac{1}{7}} + \frac{3775}{98784} f_{n+\frac{2}{7}} + \frac{21011}{370440} f_{n+\frac{3}{7}} + \frac{5329}{1481760} f_{n+\frac{4}{7}} + \frac{43}{49392} f_{n+\frac{5}{7}} - \frac{481}{1481760} f_{n+\frac{6}{7}} \right]. \quad (12)$$

While the evaluation of the first derivative of the continuous scheme at all points yields

$$y'_n = -\frac{1}{1270080h} \begin{bmatrix} 88905600 y_{n+\frac{3}{7}} - 88905600 y_{n+\frac{4}{7}} + 535153 f_n + 16832 f_{n+1} + 2766614 f_{n+\frac{1}{7}} + \\ 481503 f_{n+\frac{2}{7}} + 3075700 f_{n+\frac{3}{7}} - 878785 f_{n+\frac{4}{7}} + 490902 f_{n+\frac{5}{7}} - 137519 f_{n+\frac{6}{7}} \end{bmatrix}. \quad (13)$$

$$y'_{n+\frac{1}{7}} = \frac{1}{1270080h} \begin{bmatrix} -88905600 y_{n+\frac{3}{7}} + 88905600 y_{n+\frac{4}{7}} + 16832 f_n + 3793 f_{n+1} - 668879 f_{n+\frac{1}{7}} \\ -2308458 f_{n+\frac{2}{7}} - 1228705 f_{n+\frac{3}{7}} - 449420 f_{n+\frac{4}{7}} + 131583 f_{n+\frac{5}{7}} - 32746 f_{n+\frac{6}{7}} \end{bmatrix}. \quad (14)$$

$$y'_{n+\frac{2}{7}} = \frac{1}{1270080h} \begin{bmatrix} 88905600 y_{n+\frac{3}{7}} - 88905600 y_{n+\frac{4}{7}} + 3793 f_n + 1471 f_{n+1} - 48106 f_{n+\frac{1}{7}} + 788223 f_{n+\frac{2}{7}} + \\ 1900660 f_{n+\frac{3}{7}} + 46175 f_{n+\frac{4}{7}} + 41622 f_{n+\frac{5}{7}} - 12239 f_{n+\frac{6}{7}} \end{bmatrix}. \quad (15)$$

$$y'_{n+\frac{3}{7}} = \frac{1}{12700800h} \begin{bmatrix} 88905600h y_{n+\frac{4}{7}} - 88905600h y_{n+\frac{3}{7}} + 1472 f_n + 1393 f_{n+1} - 14639 f_{n+\frac{1}{7}} + 76182 f_{n+\frac{2}{7}} \\ -675265 f_{n+\frac{3}{7}} - 349580 f_{n+\frac{4}{7}} + 66783 f_{n+\frac{5}{7}} - 13546 f_{n+\frac{6}{7}} \end{bmatrix}. \quad (16)$$

$$y'_{n+\frac{4}{7}} = \frac{1}{12700800h} \begin{bmatrix} 88905600 y_{n+\frac{3}{7}} - 88905600 y_{n+\frac{4}{7}} + 1393h^2 f_n + 1472h^2 f_{n+1} - 13546h^2 f_{n+\frac{1}{7}} + 66783h^2 f_{n+\frac{2}{7}} \\ -349580h^2 f_{n+\frac{3}{7}} - 675265h^2 f_{n+\frac{4}{7}} + 76182h^2 f_{n+\frac{5}{7}} - 14639h^2 f_{n+\frac{6}{7}} \end{bmatrix}. \quad (17)$$

$$y'_{n+\frac{5}{7}} = \frac{1}{12700800h} \begin{bmatrix} 88905600 y_{n+\frac{4}{7}} - 88905600 y_{n+\frac{3}{7}} + 1472h^2 f_n + 3793h^2 f_{n+1} - 12239h^2 f_{n+\frac{1}{7}} + 41622h^2 f_{n+\frac{2}{7}} \\ +46175h^2 f_{n+\frac{3}{7}} + 1900660h^2 f_{n+\frac{4}{7}} + 788223h^2 f_{n+\frac{5}{7}} - 48106h^2 f_{n+\frac{6}{7}} \end{bmatrix}. \quad (18)$$

$$y'_{n+\frac{6}{7}} = \frac{1}{12700800h} \begin{bmatrix} 88905600 y_{n+\frac{3}{7}} - 88905600 y_{n+\frac{4}{7}} + 3793h^2 f_n + 16832h^2 f_{n+1} - 32746h^2 f_{n+\frac{1}{7}} \\ +131583h^2 f_{n+\frac{2}{7}} - 449420h^2 f_{n+\frac{3}{7}} - 1228705h^2 f_{n+\frac{4}{7}} - 2308458 f_{n+\frac{5}{7}} - 668879 f_{n+\frac{6}{7}} \end{bmatrix}. \quad (19)$$

$$y'_{n+1} = \frac{1}{12700800h} \begin{bmatrix} 88905600 y_{n+\frac{4}{7}} - 88905600 y_{n+\frac{3}{7}} + 16832h^2 f_n + 535153h^2 f_{n+1} - 137519h^2 f_{n+\frac{1}{7}} \\ +490902h^2 f_{n+\frac{2}{7}} - 878785h^2 f_{n+\frac{3}{7}} + 3075700h^2 f_{n+\frac{4}{7}} + 481503h^2 f_{n+\frac{5}{7}} + 2766614h^2 f_{n+\frac{6}{7}} \end{bmatrix}. \quad (20)$$

The schemes in equations (7) - (13) are combined together in matrix form and by using the matrix inversion technique, a block method of the following form is produced:

$$\begin{bmatrix} y_{n+1/7} \\ y_{n+2/7} \\ y_{n+3/7} \\ y_{n+4/7} \\ y_{n+5/7} \\ y_{n+6/7} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} y_n + \begin{bmatrix} \frac{1}{7} \\ \frac{2}{7} \\ \frac{3}{7} \\ \frac{4}{7} \\ \frac{5}{7} \\ \frac{6}{7} \\ 1 \end{bmatrix} hy'_n + h^2 \begin{bmatrix} \frac{33953}{3175200} & -\frac{341699}{29635200} & \frac{105943}{8890560} & -\frac{153761}{17781120} & \frac{943}{231525} & -\frac{99359}{88905600} & \frac{6031}{44452800} \\ \frac{27821}{694575} & -\frac{17}{675} & \frac{799}{27783} & -\frac{5881}{277830} & \frac{2321}{231525} & -\frac{1916}{694575} & \frac{233}{694575} \\ \frac{39141}{548800} & -\frac{24111}{1097600} & \frac{369}{7840} & -\frac{7299}{219520} & \frac{8613}{548800} & -\frac{4737}{1097600} & \frac{9}{17150} \\ \frac{71152}{694575} & -\frac{3832}{231525} & \frac{11344}{138915} & -\frac{856}{19845} & \frac{4912}{231525} & -\frac{4072}{694575} & \frac{496}{694575} \\ \frac{59375}{444528} & -\frac{13375}{1185408} & \frac{210625}{1778112} & -\frac{130625}{3556224} & \frac{25}{864} & -\frac{26875}{3556224} & \frac{1625}{1778112} \\ \frac{1413}{8575} & -\frac{54}{8575} & \frac{267}{1715} & -\frac{99}{3430} & \frac{459}{8575} & -\frac{9}{1225} & \frac{9}{8575} \\ \frac{25333}{129600} & \frac{49}{86400} & \frac{245}{1296} & -\frac{833}{51840} & \frac{3283}{43200} & \frac{2989}{259200} & \frac{167}{64800} \end{bmatrix} \begin{bmatrix} f_{n+1/7} \\ f_{n+2/7} \\ f_{n+3/7} \\ f_{n+4/7} \\ f_{n+5/7} \\ f_{n+6/7} \\ f_{n+1} \end{bmatrix} + h^2 f_n \begin{bmatrix} \frac{416173}{88905600} \\ \frac{14939}{1389150} \\ \frac{18399}{1097600} \\ \frac{15824}{694575} \\ \frac{102425}{3556224} \\ \frac{597}{17150} \\ \frac{10597}{259200} \end{bmatrix}. \quad (21)$$

Substituting the schemes that made up the block in (21) into equations (13) - (20), gives equations (22) – (28)

$$y'_{n+\frac{1}{7}} = y'_n + h \left[\frac{751}{17280} f_n + \frac{275}{169344} f_{n+1} + \frac{139849}{846720} f_{n+\frac{1}{7}} - \frac{4511}{31360} f_{n+\frac{2}{7}} + \frac{123133}{846720} f_{n+\frac{3}{7}} - \frac{88547}{846720} f_{n+\frac{4}{7}} + \frac{1537}{31360} f_{n+\frac{5}{7}} - \frac{11351}{846720} f_{n+\frac{6}{7}} \right]. \quad (22)$$

$$y'_{n+\frac{2}{7}} = y'_n + h \left[\frac{8}{6615} f_{n+1} + \frac{1466}{6615} f_{n+\frac{1}{7}} - \frac{71}{2940} f_{n+\frac{2}{7}} + \frac{66}{735} f_{n+\frac{3}{7}} - \frac{1927}{26460} f_{n+\frac{4}{7}} + \frac{26}{735} f_{n+\frac{5}{7}} - \frac{26}{2940} f_{n+\frac{6}{7}} + \frac{41}{980} f_n \right]. \quad (23)$$

$$y'_{n+\frac{3}{7}} = y'_n + h\left[\frac{265}{6272}f_n + \frac{9}{6272}f_{n+1} + \frac{1359}{6272}f_{n+\frac{1}{7}} + \frac{1377}{31360}f_{n+\frac{2}{7}} + \frac{5927}{31360}f_{n+\frac{3}{7}} - \frac{3033}{31360}f_{n+\frac{4}{7}} + \frac{1377}{31360}f_{n+\frac{5}{7}} - \frac{373}{31360}f_{n+\frac{6}{7}}\right]. \quad (24)$$

$$y'_{n+\frac{4}{7}} = y'_n + h\left[\frac{278}{6615}f_n + \frac{8}{6615}f_{n+1} + \frac{1448}{6615}f_{n+\frac{1}{7}} + \frac{8}{245}f_{n+\frac{2}{7}} + \frac{1784}{6615}f_{n+\frac{3}{7}} - \frac{106}{6615}f_{n+\frac{4}{7}} + \frac{6}{245}f_{n+\frac{5}{7}} - \frac{64}{6615}f_{n+\frac{6}{7}}\right]. \quad (25)$$

$$y'_{n+\frac{5}{7}} = y'_n + h\left[\frac{265}{6272}f_n + \frac{275}{169344}f_{n+1} + \frac{36725}{169344}f_{n+\frac{1}{7}} + \frac{775}{18816}f_{n+\frac{2}{7}} + \frac{4625}{18816}f_{n+\frac{3}{7}} + \frac{13625}{169344}f_{n+\frac{3}{7}} + \frac{13625}{169344}f_{n+\frac{4}{7}} + \frac{1895}{18816}f_{n+\frac{5}{7}} - \frac{275}{18816}f_{n+\frac{6}{7}}\right]. \quad (26)$$

$$y'_{n+\frac{6}{7}} = y'_n + h\left[\frac{41}{980}f_n + \frac{54}{245}f_{n+\frac{1}{7}} + \frac{27}{980}f_{n+\frac{2}{7}} + \frac{68}{245}f_{n+\frac{3}{7}} + \frac{27}{980}f_{n+\frac{4}{7}} + \frac{54}{245}f_{n+\frac{5}{7}} + \frac{41}{980}f_{n+\frac{6}{7}}\right]. \quad (27)$$

$$y'_{n+1} = y'_n + h\left[\frac{751}{17280}f_n + \frac{751}{17280}f_{n+1} + \frac{3577}{17280}f_{n+\frac{1}{7}} + \frac{49}{640}f_{n+\frac{2}{7}} + \frac{2989}{17280}f_{n+\frac{3}{7}} + \frac{2989}{17280}f_{n+\frac{4}{7}} + \frac{49}{640}f_{n+\frac{5}{7}} + \frac{3577}{17280}f_{n+\frac{6}{7}}\right]. \quad (28)$$

2.1 Investigation of the Fundamental Properties of the Method

Order and Error Constant:

(Lambert, 1973)**'s** method for finding the order of a numerical scheme is also applied to equation (21). Hence, the new hybrid block method is of equal order $\rho = 8$ with the error constants represented by the vector,

$$C_{10} = \left[-\frac{3625}{27334564895232} \quad -\frac{2119}{32552} \quad -\frac{4223}{7537} \quad \frac{7221}{8745} \quad \frac{6167}{2352} \quad \frac{6637}{78331} \quad \frac{151}{723}\right]^T$$

Zero Stability of the Block

Definition: The block is said to be zero stable if the roots $z_{s,}\, s = 1, 2, 3, \ldots, n$ of the characteristics polynomial $\rho(z)$ defined by $\rho(z) = \det(zA - E)$ satisfies $|z_s| \leq 1$ and the roots $|z_s| = 1$ is simple. See (Kuboye et al., 2022)

For our one-step hybrid method,

$$A = \left[z\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right] = 0$$

$$A = z^6(z-1) = 0, z = 0,0,0,0,0,0,0$$

Hence the block is zero stable. See (Abolarin, et al., 2020b)

Consistency

Definition: A block method is said to be consistent if its order is greater than one. Consistency property is achieved for the one-step hybrid block method from the above analysis since the order = $8 \geq 1$.

Convergence

Theorem 1. Consistency and zero stability are essential prerequisites for a linear multistep technique to be convergent, according to (Lambert, 1973). As a result, because it meets both the consistency and zero stability requirements, the developed one-step hybrid block approach, in (21) is convergent.

3.0 Results

Numerical Experiments

The performance of the one-step hybrid approach on certain test examples is investigated in this section. The results of the test cases are presented in a tabular format. For computational objectives, we used MAPLE codes.

Example 1.

We consider Stiefel and Bettis example

$$y'' + y = 0.001\cos(x), \quad y(0) = 1, \; y'(0) = 0$$
$$z'' + z = 0.001\sin(x), \qquad y(0) = 0, y'(0) = 0.9995.$$
$$h = \frac{1}{320}$$

The analytical solution of the above problem is given as

$$y(x) = \cos(x) + 0.0005x\sin(x),$$
$$z(x) = \sin(x) - 0.0005x\cos(x)$$

Table 1b Showing the Comparison of the Error for Test Example 1 with the Errors In (Adeniran, et al 2015)

X	Y Error	Z Error	Error in (Adeniran et al., 2015) for Y	Error in (Adeniran et al., 2015) for Z
1/320	9.14736E-11	2.60407E-13	8.5260E-12	2.6041E-13
3/320	2.74421E-10	3.96895E-15	2.5570E-11	9.4100E-13
6/320	5.48819E-10	2.32380E-12	5.1129E-11	2.3610E-12
9/320	8.23146E-10	6.98300E-12	7.6656E-11	4.2601E-12

12/320	1.09736E-09	1.39729E-11	1.0215E-10	6.6375E-12
15/320	1.37141E-09	2.32924E-11	1.2760E-10	9.4931E-12
18/320	1.64525E-09	3.49399E-11	1.5300E-10	1.2825E-11
21/320	1.91884E-09	4.89136E-11	1.7836E-10	1.6635E-11
24/320	2.19215E-09	6.52111E-11	2.0365E-10	2.0921E-11

Example 2: We consider a linear second order problem

$$y'' = y', \qquad y(0) = 0,\ y'(0) = -1,\ h = 0.01$$

Exact solution: $y(x) = 1 - e^{x}$

Table 2a Showing the Computed Result for Test Example 2

X	Exact Solution	Computed Solution	Error
0.1	-0.0100501670951134745600000	-0.01005016708416805754216545690	5.831942458E-12
0.2	-0.02020134003763240403602565	-0.02020134002675581016014392048	1.087659387E-11
0.3	-0.03045453396863439597511297	-0.03045453395351685561243995383	1.511754063E-11
0.4	-0.04081077421092634441660699	-0.04081077419238822675704475792	1.853811766E-11
0.5	-0.05127109639714544740269340	-0.05127109637602403969751763634	2.112140771E-11
0.6	-0.06183654656820985747238019	-0.06183654654535962222468487717	2.285023525E-11
0.7	-0.07250818127792364343102381	-0.07250818125421647905310394989	2.370716438E-11
0.8	-0.0832870676986330494607540500136	-0.08328706767495855443598775867	2.367449502E-11
0.9	-0.0941742837279446172795538881622	-0.09417428370521035787289762354	2.273425941E-11
1.0	-0.105170918096515843249964490424	-0.10517091807564762481170782649	2.086821844E-11

Table 2b Showing the Comparison of the Error for Test Example 2 with the Errors in (Adeyefa & Kuboye, 2020) and (Mohammed, 2011)

X	Error	Error in (Adeyefa & Kuboye, 2020)	Error in (Mohammed, 2011)
0.1	5.831942458E-12	2.095826E-10	2.198000000E-05
0.2	1.087659387E-11	2.092718E-09	6.070400000E-06
0.3	1.511754063E-11	7.842546E-09	1.005100000E-05
0.4	1.853811766E-11	2.009500E-08	1.402530000E-05
0.5	2.112140771E-11	4.199771E-08	1.799340000E-05
0.6	2.285023525E-11	7.728842E-08	2.161620000E-05
0.7	2.370716438E-11	1.303844E-07	2.799300000E-05
0.8	2.367449502E-11	2.064839E-07	3.456100000E-05
0.9	2.273425941E-11	3.116817E-07	4.111400000E-05
1.0	2.086821844E-11	4.531001E-07	4.765600000E-05

Example 3: We consider a highly stiff initial value problem

$$y'' = 100y, \qquad y(0) = 1,\ y'(0) = -10,\ h = 0.01$$

Exact solution: $y(x) = e^{-10x}$

Table 3 Showing the Computed Result for Test Example 3

X	Exact solution	Computed solution	Error	Error in (Kamo et al., 2018)
0.01	0.904837418035959573	0.904837418000000000	3.5959573E-11	6.6281E-11
0.02	0.818730753077981859	0.818730753012906724	6.5075135E-11	1.6280E-10
0.03	0.740818220681717866	0.740818220593394240	8.8323626E-11	2.5675E-10
0.04	0.670320046035639301	0.670320045929081272	1.06558029E-10	3.4946E-10
0.05	0.606530659712633424	0.606530659592111310	1.20522114E-10	4.7320E-10
0.06	0.548811636094026433	0.548811635963162930	1.30863503E-10	7.0419E-10
0.07	0.496585303791409515	0.496585303653264289	1.38145226E-10	9.6018E-10
0.08	0.449328964117221591	0.449328963974365626	1.42855965E-10	1.2232E-09
0.09	0.406569659740599112	0.406569659595180011	1.45419101E-10	1.4962E-09
1.00	0.367879441171442322	0.367879441025241607	1.46200715E-10	1.8005E-09

Example 4: We consider an application Problem (Mass Spring Motion)
A 128lb weight is attached to a spring having a spring constant of 64lb/ft. The weight is started in motion with no initial velocity by displacing it 6 *inches* above the equilibrium position and by simultaneously applying to the weight an external force, $F(t) = 8\sin(4t)$. Assuming no air resistance, compute the subsequent motion of the weight at $t : 0.01 \le t \le 0.10$
Consequently, we model this problem into a mathematical equation and then apply our method to compute the motion on the weight attached to the spring. The following parameters were considered.
$m = 128,\ k = 64,\ b = 0,$ and $F(t) = 0$. Thus, the application problem is written mathematically as

$$\frac{d^2x}{dt^2} + 16x = 2\sin(4t), x(0) = \frac{-1}{2},\ x'(0) = 0$$

With the analytical solution given below

$$x(t) = \frac{-1}{2}\cos(4t) + \frac{1}{16}\sin(4t) - \frac{t}{4}\cos(4t)$$

Table 4 Showing the Computed Result for Test Example 4

x	Exact solution	Computed solution	Error	Error in (Skwame et al., 2018)
0.01	- 0.499598720210476780	- 0.499598720179987850	3.04889300E-11	1.6621E-09
0.02	- 0.498390193309749496	- 0.498390193248525886	6.12236100E-11	1.1586E-08
0.03	- 0.496368369740279663	-0.496368369648142767	9.21368960E-11	2.9743E-08
0.04	- 0.493528526608179370	- 0.493528526485018636	1.23160734E-10	5.6076E-08
0.05	- 0.489867287968945010	- 0.489867287814718711	1.54226299E-10	9.0504E-08
0.06	- 0.485382642897099334	- 0.485382642711835214	1.85264120E-10	1.3291E-07
0.07	- 0.480073961290566858	- 0.480073961074362619	2.16204239E-10	1.8317E-07
0.08	- 0.473942007364361891	- 0.473942007117385560	2.46976331E-10	2.4110E-07
0.09	- 0.466988950792027839	- 0.466988950514517973	2.77509866E-10	3.0653E-07
1.00	- 0.459218375457224013	- 0.459218375149489772	3.07734241E-10	3.7922E-07

4. Discussion of Results

Table 1a shows the results of the new one-step hybrid technique applied to the Stiefel and Bettis second order ODE (test example 1). The conclusions achieved by our method competes favourably with those reached by (Adeniran et al., 2015) as shown in Table 1b. The results of our technique when applied to linear second order ODEs are shown in Table 2a (test example

2). When compared to their errors, our proposed system outperforms the methods of (Adeyefa and Kuboye, 2020) and (Mohammed, 2011). The results of the error comparison are shown in Table 2b. In Table 3, we considered and solve a highly stiff problem and compared the result with (Kamo et al., 2018). The new method gives a minimal error. Furthermore, we also solve an application problem in physical sciences and engineering namely Mass Spring Motion in other to examine the usefulness and applicability of the new method. We present the numerical results and comparison of the absolute error with (Skwame et al., 2018) using the same values of h in tables 4a and 4b respectively. It is very obvious that the new method is advantageous over other existing methods who solved the same problems in the literatures.

5. Conclusion

This study develops a one-step implicit hybrid block technique for numerical solution of second order ODEs with initial value problems. The new method developed via interpolation and collocation techniques, is of order, $\rho = 8$. It was found to be zero-stable, consistent, and convergent. It also gives precise results that outperform existing methods in the literature in terms of absolute error. The performance of the new method was found when the method was applied on some second order IVP which includes Stiefel and Bettis equation, linear and stiff second order IVP and a real-life application problem arising from mass spring motion. The major contributions to knowledge from this research are: (i) a new one-step hybrid block method with six hybrid points was developed (ii) the method satisfies all conditions for the basic properties of a linear multistep method (iii) the method solves second order IVP directly thereby removing the stress and setbacks associated with the reduction approach and (iv) the developed method produced more accurate results than current methods when used to solve some second order IVP.

Acknowledgements

The authors wish to appreciate the editor and the anonymous reviewers for their valuable comments and suggestions which have further improved the quality of this paper.

Declaration of conflict of interest

The authors have collectively contributed to the conceptualization, design, and execution of this journal. They have worked on drafting and critically revising the article to include significant intellectual content. This manuscript has not been previously submitted or reviewed by any other journal or publishing platform. Additionally, the authors do not have any affiliation with any organization that has a direct or indirect financial stake in the subject matter discussed in this manuscript.

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